EXAMINING THE DEVELOPMENT OF A FOURTH GRADE STUDENT AS SHE ENGAGES IN REASONING AND PROVING

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In this study, we investigated one student as she engaged in the exploration of a graph theory problem, the Graceful Tree Conjecture. The study took place in a classroom in the Midwestern United States which contained eighteen students. We report on a descriptive case study of one student, Heidi, as she works her way through different classes of tree graphs and advances through different types of reasoning, justifications, and proving a select class of tree graphs can be labeled gracefully. We found that with repeated experiences, Heidi, was able to advance her level of reasoning and creations of justifications.

Keywords: Problem Solving, Reasoning and Proof, Elementary School Education

The current horizon in mathematics education for elementary school focuses on procedural and conceptual understanding of mathematical content (Bieda, Ji, Drwencke, & Picard, 2013). What would happen if we went against the horizon and instead focused on the mathematical processes of reasoning and proving?

Students engaging in mathematical proof is not characteristically presented until middle school (Lin & Tsia, 2016) or even high school (Stylianides, 2007). Students at the elementary level are typically focused on finding the correct answer but justifying their solutions is not included (Kieran, 2004). Carpenter, Franke, and Levi (2003) have identified three ways students tend to give an argument or mathematical justification: (a) appeal to authority, (b) justification by example, and (c) generalizable argument. A student appealing to authority would give a reason for their answer by stating a rule or procedure their teacher has shared. A justification through example would be where a student makes a case about something through sharing a specific case, such as four times two is eight therefore an even multiplied by an even is always even. For a generalized argument students would give an argument that would be applicable to all cases in the given conjecture.

Carpenter et al. (2003) furthered their team's argument by stating that elementary students do not typically complete generalizable arguments but as they advance in grade level they should be encouraged to develop more generalized arguments and understand how just giving an example restricts their argument. Past researchers (Ball, 1993; Keith, 2006; Lin & Tsai 2016) have documented that elementary students are able to make conjectures and develop justifications. This has been done through students in second and third grade exploring the idea of the sums of even and odd numbers through using blocks and definitions. Many researchers have recommended all elementary students engage in argumentation, proof, and making mathematical justifications in all mathematics content areas (e.g., Ball & Bass, 2003; Carpenter et al., 2003; NCTM 2014; Stylianides, 2007). Bieda, Ji, Drwencke, and Picard (2013) argued students' difficulties with learning to construct formal proofs in geometry high school classes and college level mathematics could be because of the lack of experiences elementary and middle schools students have had with justifying, reasoning and proving. Further, Beida et al. (2013) stated that tasks where elementary students could engage in reasoning and proving is an area of needed

research. For this study, we examined a fourth grade student's development in reasoning and proving as she worked through a graph theory problem.

Research Questions

The following questions guided our research study:

1. What ways does a fourth grade student attempt to construct an argument or mathematical justification?
2. How does a fourth grade student progress through repeated opportunities with justifying and proof?

Theoretical Framework

Throughout this study, we wanted students to experience mathematics more similar to how a mathematician might experience it than what they routinely see in a traditional elementary school setting. Thus, we tried to create a community of practice where students had the freedom to understand a mathematical topic through exploration and focused on the topic of reasoning and proving. Because of this, we used the idea of communities of practice (Wenger, 1998) as an overarching theoretical framework for this study and attempted to turn the fourth grade classroom into such an environment.

We used communities of practice as a way to create an experience among the students that would be similar to the ways a mathematician might engage in mathematics. In communities of practice the two major components are practices and identities. This idea develops from the thought that we are "active participants in the *practices* of social communities and constructing *identities* in relation to these communities" (Wenger, 1998, p. 4). The main idea of practice is how we experience the world and create social engagement. There are three components of the relationship between practice and the created community, which are mutual engagement, joint enterprise, and shared repertoire. We attempted to create mutual engagement by allowing the students opportunities to work together; joint enterprise by allowing the students to create a similar goal of finding a solution to the graph theory problem; and shared repertoire by teaching the students new words such as node, edge, conjecture, and the idea of a graceful labeling to use while working in a community of mathematicians. The other main component of communities of practice is identity. According to Wenger (2008), identity is the relationship between the personal and social and thus it allows it to shape a person's belonging in the community.

Methods

In this qualitative study, we investigated fourth grade students while they engaged in the exploration of a graph theory problem, the Graceful Tree Conjecture. The study took place in a classroom containing eighteen students at a private school in the Midwestern United States. All eighteen students participated in the study but because of the length of the paper, we will report on only one student, who we will call Heidi. The students participated in three teaching experiments (Steffe & Thompson, 2000), each lasting 75 minutes. Both of the authors were teacher-researchers in this teaching experiment but not the regular classroom teacher.

During the teaching experiment, the students engaged in parts of the Graceful Tree Conjecture. The used tasks come from a previous research study (O'Dell, 2017). This study looks to extend that research by integrating the ideas of reasoning and proof into a classroom setting.

Graceful Tree Conjecture

The Graceful Tree Conjecture is currently an unsolved problem from graph theory that is accessible to elementary school students. The problem has children explore different types of tree graphs, graphs that are connected with no cycles (see Figure 1). A tree graph must be acyclic which means if you follow along the edges from node to node, you will never cycle back to the same node without retracing an edge. All tree graphs contain one more node than edge.

To create a graceful label for a tree graph, you assign numbers then create a labeling for the edges. For a tree graph of order *m*, every node is labeled distinctly from 1 through m and the edges are then labeled with the absolute value of the difference of the labels on their attached nodes. A tree graph is labeled gracefully if the edges are labeled 1 through *m*-1 distinctly (see Figure 1).

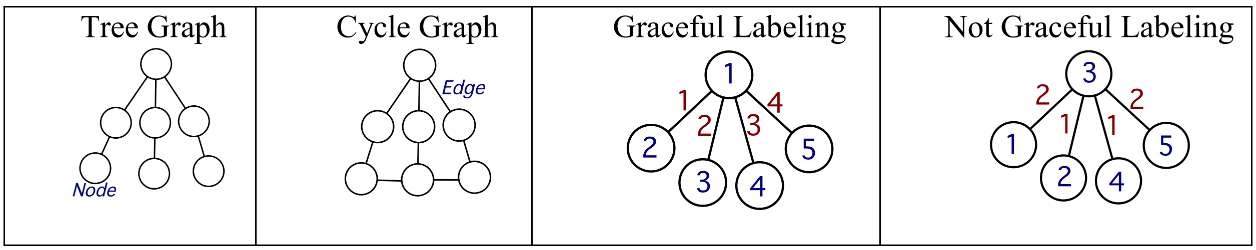

Figure 1: Graceful Tree Concepts

Overview of Teaching Experiment

During the three teaching experiment sessions, students explored different classifications of tree graphs in increasing sophistication (see Figure 2). We tasked students to not just create a graceful labeling for each graph but to find patterns, create justifications, and prove that any graph in the given category could be labeled gracefully. During the exploration of each graph classification, students were given a page that contained the first four distinct tree graphs in a given class, enlarged copies of each of the graphs, and numbered circle and square chips (see Figure 3). The use of the enlarged graphs and chips allowed the students to try multiple configurations without having to erase in hopes of easing frustration.

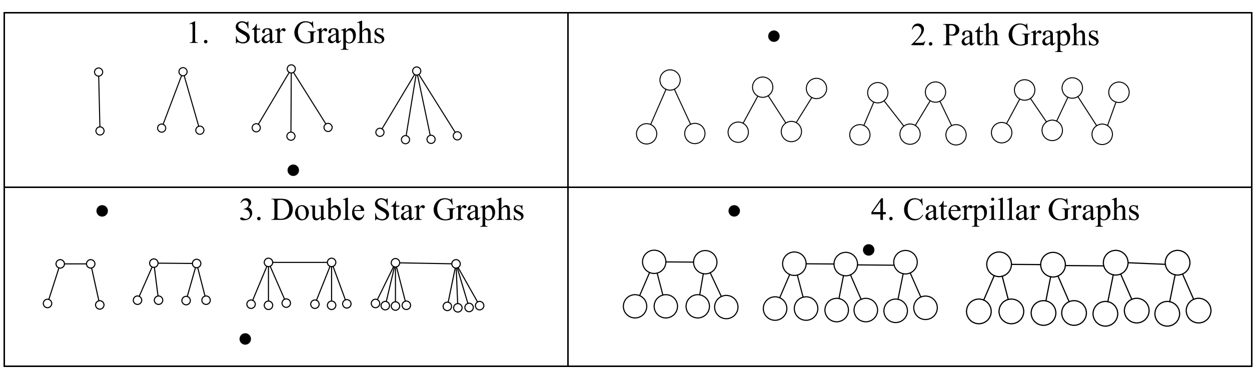

Figure 2: Different Classes of Tree Graphs in Increasing Sophistication

During the first class period students were taught what a conjecture was, each subsequent session started with a reminder that we are looking for a conjecture and what that means. Students were introduced to graph theory and tree graphs, explored graceful labelings, and were

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tasked to create a graceful labeling of the Star class. Throughout the second class period, students worked to develop a pattern or justification for path graphs. In the third session, students explored and developed justifications for graceful labelings for the group of graphs in the Double Star class and Caterpillar class. At the end of each class, we would have a whole group discussion about the different patterns, justifications, and ideas for proving that a certain class of graphs could be labeled gracefully. For every classification of tree graph students were able to generate at least one generalizable solution. During all three sessions, we encouraged the students to work together, giving them only enough hands-on materials for every two people which meant they would have to work with a partner but record solutions on their own page.

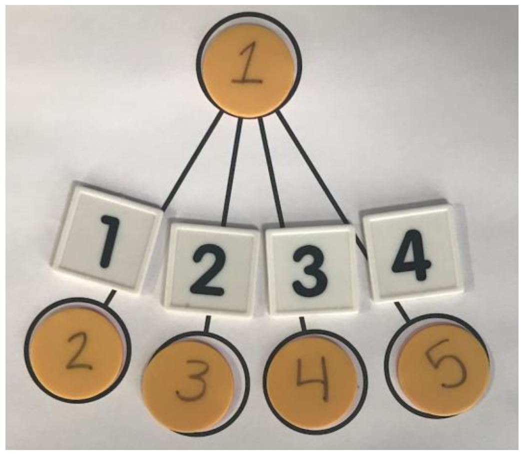

Figure 3: Enlarged Star Graph and Circle and Square Numbered Chips

Results and Discussion

All of the eighteen fourth grade students were able to make progress on each class of tree graphs. They were able to find graceful labelings for the first four distinct graphs in each of the first three classes of tree graphs, Stars, Path, and Double Star Graphs. However, not all of the fourth grade students were able to find patterns and create justifications for each of the graphs. Heidi was selected to be the case study student because she was able to create patterns for all four classes of tree graphs. Many other students were also able to do this for several of the different classes but in the given time period Heidi was the only one to find a pattern for the Caterpillar class. We had to end the session prior to the other students completing this. Heidi was also selected because she sincerely enjoyed the activity and was very willing to share her solutions with the researchers and her classmates during the class discussion period.

Star Graphs

To begin Star Graphs, Heidi quickly worked through labeling the first four distinct graphs in the class (see Figure 2 for the first four distinct graphs). Then she said she did not understand what to do next. We told her to draw and produce the next graph in the Star class by asking her what the next graph would look like. She asked, “Can I draw it?” and we told her yes. Her partner asked if they needed to have six nodes and she responded incorrectly, “You only need five.” She realized her error then changed her answer to needing six nodes. Like the first four graphs, she placed one at the top and put the rest of the numbers in order.

After she drew a graceful labeling for the six noded graph, she asked what to do next. We asked her, “Do you think you could gracefully label any Star graph?” Heidi did not seem to understand the question. We clarified that if there was a Star graph with so many nodes you were

not sure how many, could you still label it gracefully. Heidi responded, “Oh.” Again, we clarified, “What if there were thirty nodes?” Heidi stated, “You would put a one at the top and go in a pattern along the bottom.” We asked her to write about it.

To generalize the Star Graphs, Heidi, wrote a description on her page and used her example of a graph with six nodes (see Figure 4). Her description said, “You would put one at the top node and then two, then three and so on depending on how many nodes there are. You label edges by subtracting the number in the nodes also edges labels go in order.”

Heidi was able to find and describe a pattern to label any Star Graph gracefully. In her description of the pattern, she limited herself by drawing a picture of one distinct graph to explain how her pattern worked. She attempted to generalize but this was her first time being pushed to go beyond just developing a solution. However, she did share that she thought all graphs in the Star class could be labeled gracefully following her given pattern.

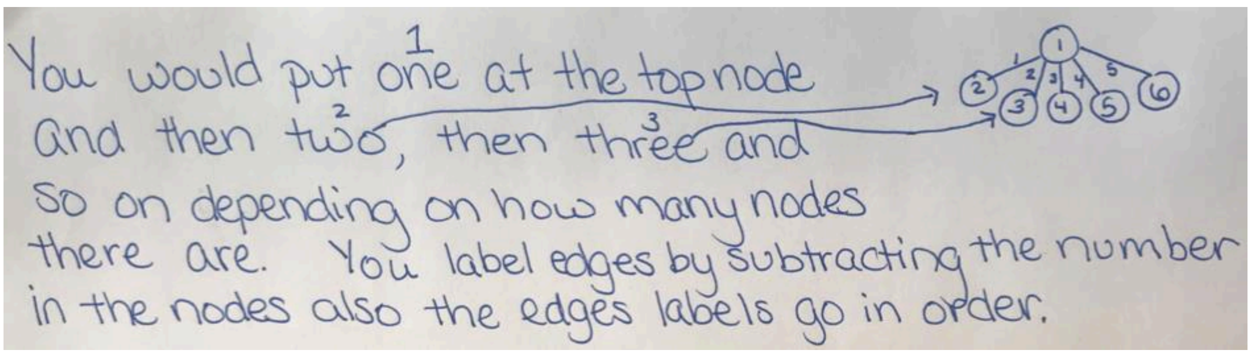

Figure 4: Heidi’s Justification for Star Graphs

Path Graphs

Heidi began working on the Path graphs and labeled the first four distinct graphs in the class. Next, she and her partner talked about what they found. Her partner said that she had the biggest number on top. Heidi said, “No, we put the smallest number on top.” When she was questioned to tell more she said, “The biggest second.” We asked her to write about her pattern. She wrote, “We used one in the first node and then the biggest number in the second node. We also used two in the third node. Then counted down every other node.”

After she finished writing about her pattern, we asked if she would use her pattern to do a Path graph with 50 nodes. She said, “Honestly, yeah.” Heidi went and began drawing and labeling a Path graph where she placed one if the first node, a fifty in the second node, a two in the third node and continued doing she had drawn eleven nodes.

Next, one of the researchers asked her what she would do if she did not know how many nodes. She said, “One would go in the top.” They asked, “What would go next?” She said, “It would be the biggest number.” Next, the researcher asked, “What would go after that?” Heidi looked at her pattern on the other side of her page and said, “Two. And then it kind of depended on…” She stopped talking and looked at her pattern. After several seconds the researcher asked, “Well you had one, the biggest number, two, and what happened with the next number?” Heidi stated, “The second biggest and yeah we did that every time.”

Then Heid went and drew the picture to give a justification, or beginning of a proof, for how she would label any tree graph in the Path class gracefully (see Figure 5). In her justification she completed how she could use her pattern for any Path graph and explained how to label the

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edges. We saw this as an advancement from her previous explanation of the Star graph where she just described a pattern and referred to one example picture. In this case, Heidi gave a complete justification or what we would consider a proof for a fourth grade student.

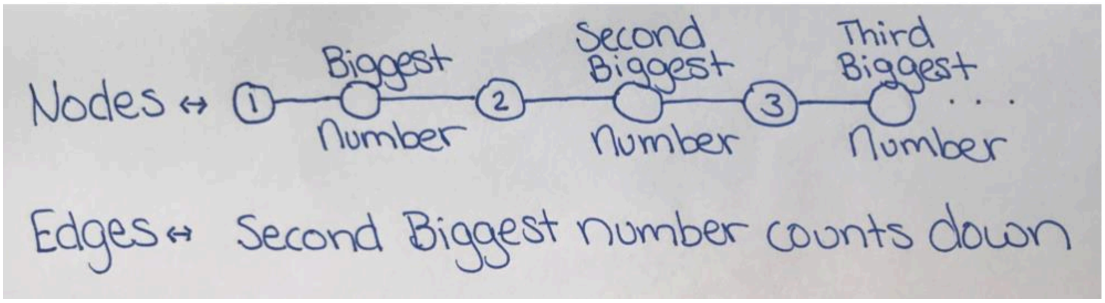

Figure 5: Heidi's Justification for Path Graphs

Double Star Graphs

Heidi began working on Double Star Graphs during the third session. Her partner from the previous two days was absent so she decided to work by herself. After gracefully labeling the first four distinct graphs, she began to write about her pattern. However, on this graph she went back to the idea of using one graph to show her pattern (see Figure 6). She went and wrote how to label the nodes. She wrote "Example Graph." She then wrote "I start the one on the top node. Then take the biggest number and put it below. It goes in a pattern too. From Biggest to smallest."

After she finished, one of the researchers came and she showed them her pattern. The researcher attempted to further the idea with trying to figure out how to label the edges. After a brief discussion they left her with the question of what would the edge that connected the two Stars of the graph might be. Heidi decided that she needed to do a few more graphs to figure it out. She went and drew a Double Star Graph with twelve nodes (see Figure 6). After thinking for a while and looking at her previous graphs in the Double Star class, she furthered this pattern by describing "The top edge is always half of the biggest number. There for whatever the number for the edge is the node is one bigger."

While Heidi was able to find a way to label any graph in the Double Star class she went back to the idea of giving an example graph to explain her pattern. We found this similar to how she labeled her Star Graph and was a slight decline in her advancement of justifications and proofs. However, she did have a more advanced thinking about the graphs than any of her classmates by finding the idea that the edge connecting the two stars of the Double Star Graph was always going to be half of the biggest number and that the node connected to the edge would be one more than half the biggest number.

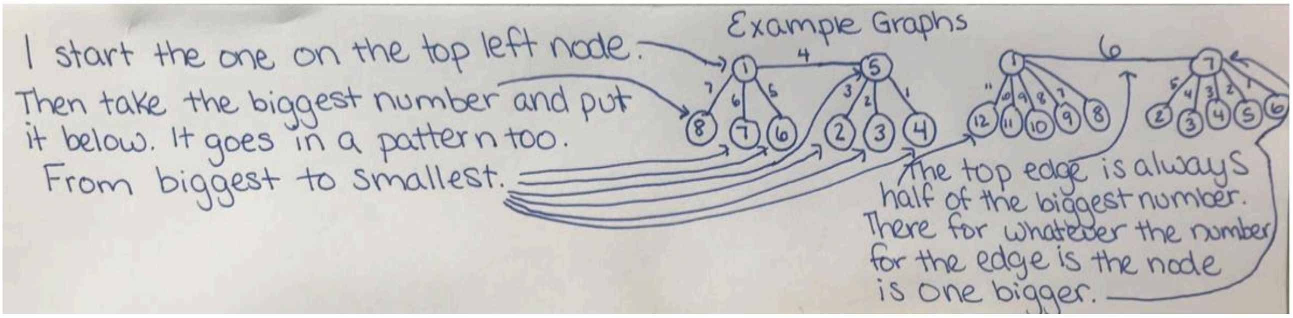

Figure 6: Heidi's Justification for Double Star Graphs

Caterpillar Graphs

Heidi asked if she could start on the next type of graph while her classmates were still finishing the previous graphs during the third session for which she had already completed her pattern. We gave her the Caterpillar Graphs on which to work. She began by completing the graphs with six, nine, and twelve nodes or the first three distinct graphs of Caterpillar class (see Figure 2).

She then drew the next graph in the class, a 15 node graph, and found a graceful labeling. She then went on to write about how she would label any graph in the Caterpillar class. She said, "I always put the one at the top far right node then put the biggest number on the bottom far right node and the second biggest number on the node next to it."

She continued by drawing a picture for how she would label any graph in this type of Caterpillar class (see Figure 7). However, this time she did not do an example of what she wrote as she had previously done for the Star Graph and the Double Star graph; rather, she completed her pattern by drawing a picture where she used the idea of biggest, second biggest, third biggest and so on. To show that her pattern would continue on, she used an ellipsis so show it would continue. She explained this idea as "So on." On this graph, Heidi did not explain how to label the edges but the researcher asked her how she labeled her edges and she verbally said they counted down in order from left to right.

Heidi showed an advancement again in her development on proof and justifications. She was able to complete a pattern and draw how to label any graph in this class similar to how she did for Path graphs.

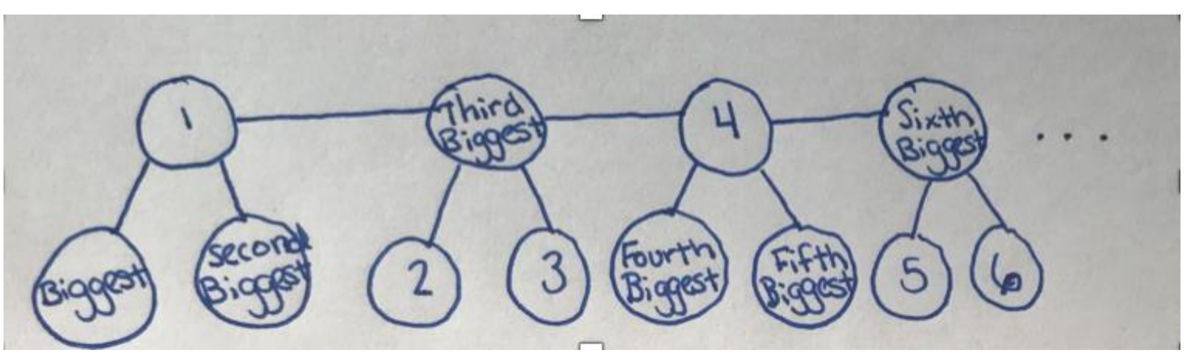

Figure 7: Heidi's Justification for Caterpillar Graphs

Conclusions

Carpenter et al. (2003) stated there are three ways students present a justification or argument, namely: appeal to authority, use an example, or give a generalized argument. We set up the situation so the fourth grade students were not able to appeal to authority because graph theory was not something they had prior knowledge of nor did the classroom teacher. The students tried to present a justification with an example. For Star Graphs, Heidi created an example to explain her pattern. She did this again on Double Star Graphs. However, on Path Graphs and Caterpillar Graphs, Heidi created generalized arguments for each. She even was able to include the ellipsis to show the pattern would continue indefinitely.

As a fourth grade student, Heidi developed through this experience from explaining how a pattern could be labeled through an example and words to creating a generalized argument for graphs in the second type of Caterpillar class. While she did not use the idea of a variable, which would not be common among fourth grade students, she was able to use terminology like the biggest number, second biggest number and so on. Further, on her first attempt at Caterpillar Graphs, she was able to discover that an edge labeling on a graph would be half of the biggest node whereas in previous graphs she only described biggest, second biggest, … numbers. In addition to finding success, Heidi, stated that she enjoyed the challenge of the activity and wondered if we would give her extra graphs to gracefully label.

This study could potentially influence the horizon of teaching because students are not typically given the chance to explore an unsolved graph theory problem in an elementary classroom. Beyond that, they are not typically given a chance to reason through an experience where they have to provide justification or proof. With the case of Heidi, she developed a more in depth understanding of what it meant to prove something with four opportunities. These types of experiences are important because students have the belief they should be able to finish a mathematics problem in five minutes or less (Schoenfeld, 1992) and we need students to understand that any good problem solving will take much longer than five minutes! Further research is needed in this area of discovery; what could happen if we went against the current horizon and shifted our expectations as a community of mathematics learners from a focus on conceptual and procedural understanding to a focus on proving, reasoning, and providing justifications for work?

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